

## TWO CHARACTERIZATIONS OF ELLIPSOIDAL CONES

JESÚS JERÓNIMO-CASTRO AND TYRRELL B. MCALLISTER

**ABSTRACT.** We give two characterizations of cones over ellipsoids. Let  $C$  be a closed pointed convex linear cone in a finite-dimensional real vector space. We show that  $C$  is a cone over an ellipsoid if and only if the affine span of  $\partial C \cap \partial(a - C)$  has dimension  $\dim(C) - 1$  for every point  $a$  in the relative interior of  $C$ . We also show that  $C$  is a cone over an ellipsoid if and only if every bounded section of  $C$  by an affine hyperplane is centrally symmetric.

## 1. INTRODUCTION

The following fact is an easy exercise in geometry: If  $E$  is an  $n$ -dimensional solid ellipsoid and  $a$  is a vector, then  $\partial E \cap \partial(a - E)$  is contained in an affine hyperplane unless  $E = a - E$ . A far more difficult result due to P. R. Goodey and M. M. Woodcock [5] shows that this property suffices to characterize ellipsoids: Ellipsoids are the only convex bodies whose boundaries have a “flat” intersection with all non-coincident translates of their negatives. Another characterization of ellipsoids is the famous False Centre Theorem of P. W. Aitchison, C. M. Petty, and C. A. Rogers [1]. This result, first conjectured by Rogers in [12], states that a convex body  $K$  ( $\dim K \geq 3$ ) is an ellipsoid if there is a point  $p$ , not a center of symmetry of  $K$ , such that every section of  $K$  by a hyperplane through  $p$  is centrally symmetric. Gruber and Ódor [6] exhibit another sense in which sufficiently symmetric convex bodies must be ellipsoids: If  $K$  is a convex body such that the cone over  $K$  from every point outside of  $K$  is symmetric about some axis, then  $K$  is an ellipsoid.

We prove analogous results for convex cones. Let us begin by fixing our notation and terminology. Let  $V$  be a finite-dimensional real vector space. A *convex linear cone* in  $V$  is a nonempty convex subset  $C \subset V$  such that  $a \in C$  and  $\lambda \geq 0$  implies that  $\lambda a \in C$ . The cone  $C$  is *pointed* if there exists a hyperplane  $H$  in  $V$  such that  $H \cap C = \{0\}$ . Henceforth, we simply write “cone” for “closed pointed convex linear cone”, unless otherwise specified.

Given a convex subset  $K \subset V$ , let  $\text{aff}(K)$  denote the affine span of  $K$ . We write  $\text{int}(K)$  and  $\partial K$  for the interior and boundary, respectively, of  $K$  relative to  $\text{aff}(K)$  under the subspace topology. The *cone over*  $K$ , denoted  $\text{cone}(K)$ , is the intersection of all cones containing  $K$ . A *section* of  $K$  is a  $(\dim K - 1)$ -dimensional intersection of  $K$  with an affine hyperplane. A *(solid) ellipsoid* in  $V$  is the image of the closed unit ball in some Euclidean vector space  $E$  under an affine map  $E \rightarrow V$ . An *ellipse* is a 2-dimensional ellipsoid. A cone  $C$  is *ellipsoidal* if some section of  $C$  is an ellipsoid.

**Definition 1.1** (FBI and CSS Cones). Let  $C \subset V$  be a cone. We say that  $C$  satisfies the *flat boundary intersections (FBI) property* if, for each  $a \in \text{int}(C)$ , the

---

2010 *Mathematics Subject Classification.* 52A20, 53A07.

*Key words and phrases.* ellipsoidal cone, centrally symmetric convex body.

affine span of  $\partial C \cap \partial(a - C)$  has dimension  $\dim(C) - 1$ . We say that  $C$  satisfies the *centrally symmetric sections (CSS) property* if every bounded section of  $C$  is centrally symmetric. We call a cone with the FBI (respectively, CSS) property an *FBI* (respectively, *CSS*) *cone*.

It is easy to check that every finite-dimensional ellipsoidal cone satisfies the FBI property: Let  $C$  be an  $(n+1)$ -dimensional cone over an ellipsoid. Then there exists a linear system  $x = (x_0, \dots, x_n)$  of coordinates on the linear span of  $C$  such that  $C$  is the set of solutions to

$$x_0^2 \geq x_1^2 + \dots + x_n^2, \quad x_0 \geq 0.$$

Fix a point  $a = (a_0, \dots, a_n) \in \text{int}(C)$ , and let  $\bar{a} = (-a_0, a_1, \dots, a_n)$ . Then  $\partial C \cap \partial(a - C)$  is contained in the affine hyperplane of solutions to the linear equation  $\bar{a} \cdot x = \frac{1}{2}\bar{a} \cdot a$ .

Our first main result is that the only finite-dimensional cones satisfying the FBI property are the ellipsoidal cones.

**Theorem 1.2** (proved on p. 4). *A cone  $C \subset V$  is an FBI cone if and only if  $C$  is an ellipsoidal cone.*

*Remark 1.3.* We make a short digression to note that Theorem 1.2 provides a natural motivation for the use of a Lorentzian inner product in special relativity. One way to develop special relativity begins as follows: Let  $M$  be a 4-dimensional real affine space of *events* with associated vector space  $V \cong \mathbb{R}^4$ . Fix a 4-dimensional cone  $C \subset V$ , called the *light cone*. At this stage, we do not yet assume that  $C$  is ellipsoidal. Given a pair of events  $a, b \in M$  with  $b - a \in \text{int}(C)$ , define the *inertial reference frame*  $\overline{ab}$  to be the set of affine lines in  $M$  parallel to the affine span of  $\{a, b\}$ .

Traditionally, one proceeds by assuming that  $\partial C$  is (one half of) the null cone of a Lorentzian inner product on  $V$ . This is equivalent to assuming that  $C$  is ellipsoidal. We then associate to a given inertial reference frame  $\overline{ab}$  a decomposition of  $V$  into a direct sum of a 3-dimensional “space” summand, parallel to the affine span of  $\partial(a + C) \cap \partial(b - C)$ , and a 1-dimensional “time” summand, spanned by  $b - a$ .

However, instead of assuming that  $C$  is ellipsoidal, we may instead proceed from the assumption that every choice of inertial reference frame yields a decomposition of  $V$  into a 3-dimensional space summand and a 1-dimensional time summand in the manner just described. In other words, we may assume that the affine span of  $\partial(a + C) \cap \partial(b - C)$  is 3-dimensional. It then follows from Theorem 1.2 that the light cone is ellipsoidal, so that a Lorentzian inner product arises naturally. Thus we derive a Lorentzian inner product on spacetime from the phenomenologically immediate datum that space is 3-dimensional.

Since every section of an ellipsoidal cone is an ellipsoid, it is clear that ellipsoidal cones are CSS cones. Our second main result is that the ellipsoidal cones are precisely the CSS cones.

**Theorem 1.4** (proved on p. 5). *A cone  $C \subset V$  is a CSS cone if and only if  $C$  is an ellipsoidal cone.*

We call Theorem 1.4 the False Centre Theorem for Cones, by analogy with the famous False Centre Theorem characterizing ellipsoids [1]. The special case

of Theorem 1.4 in which  $C$  is a 3-dimensional cone was proved in [11]. A more-recent independent proof of this 3-dimensional case appeared in the B.S. thesis of Efrén Morales-Amaya [10]. Solomon [13] shows that any complete connected  $C^2$  surface in  $\mathbb{R}^3$  whose sections are all centrally symmetric ovals is either a cylinder or a quadric. We note that neither Theorem 1.2 nor Theorem 1.4 relies on any smoothness assumptions on the boundary of the cone, though convexity will be crucial for our proofs.

The outline of our argument is as follows. In Section 2, we use the previously established result that 3-dimensional CSS cones are ellipsoidal (*i.e.*, the 3-dimensional case of Theorem 1.4) to prove that  $n$ -dimensional FBI cones are ellipsoidal (Theorem 1.2). Then, in Section 3 we use Theorem 1.2 to prove Theorem 1.4 for cones of arbitrary finite dimension.

## 2. FBI CONES ARE ELLIPSOIDAL CONES

We begin with a few straightforward lemmas about the intersection of the boundary of a cone  $C$  with the boundary of a translation of  $-C$  by a vector in the interior of  $C$ .

**Lemma 2.1.** *Suppose that  $C \subset V$  is a cone and that  $a \in \text{int}(C)$ . Then  $\partial C \cap \partial(a - C)$  is centrally symmetric about  $\frac{1}{2}a$ .*

*Proof.* The translation  $\partial C \cap \partial(a - C) - \frac{1}{2}a = \partial(C - \frac{1}{2}a) \cap \partial(-C + \frac{1}{2}a)$  is centrally symmetric about the origin, so the original intersection is centrally symmetric about  $\frac{1}{2}a$ .  $\square$

**Lemma 2.2.** *Suppose that  $C \subset V$  is a cone and that  $a \in \text{int}(C)$ . Let  $\Gamma := \partial C \cap \partial(a - C)$  and let  $S$  be a bounded section of  $C$ . Then every point on  $\Gamma$  (resp.  $\partial S$ ) is a unique scalar multiple of a unique point on  $\partial S$  (resp.  $\Gamma$ ). Moreover, this correspondence  $\partial S \leftrightarrow \Gamma$  is a homeomorphism, so that  $\Gamma$  is homeomorphic to an  $n$ -sphere.*

*Proof.* Fix a bounded section  $S$  of  $C$ . Then every point on  $\partial C \setminus \{0\}$  is a unique scalar multiple of a unique point on  $\partial S$ . In particular, we have a map  $\Gamma \rightarrow \partial S$ .

To establish the converse correspondence  $\partial S \rightarrow \Gamma$ , fix  $\lambda > 0$  so that  $\lambda a \in S$ . Given  $x \in \partial S$ , let  $r(x)$  be the opposite endpoint of the chord of  $S$  through  $\lambda a$  starting at  $x$ . Let  $\mu_x \in (0, 1/\lambda)$  be such that  $\lambda a = \lambda \mu_x x + (1 - \lambda \mu_x) r(x)$ . On the one hand,  $\mu_x > 0$  and  $x \in \partial C$  imply that  $\mu_x x \in \partial C$ . On the other hand,  $\mu_x < 1/\lambda$ ,  $r(x) \in \partial C$ , and  $\mu_x x = a - (1/\lambda - \mu_x) r(x)$  together imply that  $\mu_x x \in \partial(a - C)$ . Hence,  $\mu_x x \in \Gamma$ . Since  $a - C$  is a pointed affine cone containing the origin in its interior,  $\mu_x x$  is the unique multiple of  $x$  on  $\partial(a - C)$ . Finally, observe that  $x \mapsto \mu_x x$  is a continuous map with a continuous inverse, establishing that  $\partial S$  and  $\Gamma$  are homeomorphic.  $\square$

**Lemma 2.3.** *Let  $K$  be a 2-dimensional convex body. Then  $K$  contains an inscribed parallelogram with vertices in  $\partial K$ .*

*Proof.* It is easy to construct such a parallelogram using the intermediate value theorem and continuity of the boundary of  $K$ . For example, consider the family  $F$  of chords perpendicular to a fixed diameter of  $K$ . Choose two chords  $\chi_1, \chi_2 \in F$  that are of equal length and that are on opposite sides of a chord of maximum length in  $F$ . Then  $P := \text{conv}(\chi_1 \cup \chi_2)$  is an inscribed parallelogram in  $K$ . Indeed,

the stronger claim that  $K$  contains an inscribed *square* is a classical result; see, e.g., [4].  $\square$

We remark that the natural generalization of Lemma 2.3 to higher dimensions does not hold. There exist convex bodies in dimension  $n \geq 5$  that do not contain inscribed parallelepipeds [9].

We will also appeal to the following classical characterization of ellipsoids due to Brunn [2]; see also [3, Lemma 16.12, p.91].

**Theorem 2.4.** *Let  $n \geq 3$  and let  $K$  be an  $n$ -dimensional convex body such that every section of  $K$  is an ellipsoid. Then  $K$  itself is an ellipsoid.*

The key additional result on which the proof of Theorem 1.2 depends is the following characterization of 3-dimensional cones over ellipses, originally due to Olovjanischnikoff [11]. See also [10].

**Theorem 2.5** (False Centre Theorem for 3-dimensional cones). *If  $C \subset V$  is a 3-dimensional CSS cone, then  $C$  is a cone over an ellipse.*

We are now ready to prove the main result of this section, that all finite-dimensional FBI cones are ellipsoidal cones (Theorem 1.2).

*Proof of Theorem 1.2 (stated on p. 2).* Let  $C$  be an  $(n+1)$ -dimensional FBI cone. Without loss of generality, we suppose that  $C$  is full-dimensional. We begin by proving the  $n = 2$  case. The case where  $n \geq 3$  will then follow by induction.

Suppose that  $\dim(C) = 3$ . Fix a bounded section  $S$  of  $C$ , and let  $H$  be the affine span of  $S$ . By Lemma 2.3, there exists a parallelogram  $P$  with vertices in  $\partial S$ . Let  $p$  be the intersection of the diagonals of  $P$ , let  $a := 2p$ , and let  $\Gamma := \partial C \cap \partial(a - C)$ . Observe that the vertices of  $P$  are contained in  $\Gamma$  because  $P$  is fixed under inversion through  $p$ . By hypothesis,  $\Gamma$  is contained in some plane  $H'$ , so we also have  $P \subset H'$ . Since  $P$  is contained in a unique hyperplane, we have that  $H = H'$ . By Lemma 2.2,  $\Gamma$  is a curve homeomorphic to a circle and contained in  $\partial S = \partial C \cap H$ . It follows that  $S$  is the convex hull of  $\Gamma$ , which, by Lemma 2.1, is centrally symmetric. Therefore,  $C$  is a 3-dimensional CSS cone and hence, by Theorem 2.5, is a cone over an ellipse.

We proceed by induction. Suppose now that  $\dim(C) = n + 1$  for  $n \geq 3$ . Fix a bounded section  $S$  of  $C$ , let  $K$  be a section of  $S$ , let  $D := \text{cone}(K)$ , and let  $L$  be the linear span of  $D$ . Fix a point  $a$  in the relative interior of  $D$ , and let  $\Gamma := \partial C \cap \partial(a - C)$ . On the one hand,  $\partial D \cap \partial(a - D)$  is contained in  $L$ . On the other hand,  $\partial D \cap \partial(a - D)$  is a subset of  $\Gamma$ , which, since  $C$  is an FBI cone, is contained in some  $n$ -dimensional hyperplane  $H$ . Note that  $H$  is not equal to  $L$ , since  $H$  has a bounded intersection with  $C$  while  $L$  does not. Hence  $\partial D \cap \partial(a - D)$  is contained in the intersection of two distinct  $n$ -dimensional hyperplanes, so  $\partial D \cap \partial(a - D)$  is contained in some  $(n - 1)$ -dimensional affine subspace. That is,  $D$  is an  $n$ -dimensional FBI cone and hence is ellipsoidal. In particular,  $K$  is an ellipsoid, which, by Theorem 2.4, implies that  $S$  is an ellipsoid, proving the claim.  $\square$

### 3. CSS CONES ARE ELLIPSOIDAL CONES

Our proof of Theorem 1.2 relied on the False Centre Theorem for 3-dimensional cones. It is natural to ask whether a False Centre Theorem holds for cones of arbitrary dimension. So far as we know, such a generalization of Theorem 2.5 has not

appeared in the literature. In this section, we use the FBI characterization of ellipsoidal cones (proved in Section 2) to prove that the CSS property also characterizes ellipsoidal cones of arbitrary finite dimension.

A well-known result in convexity states that every point in the interior of a convex body is the centroid of some section of that body:

**Theorem 3.1** ([14]; see also [7]). *Let  $K \subset V$  be a convex body, and let  $p \in \text{int}(K)$ . Then there exists a section  $S$  of  $K$  such that  $p$  is the centroid of  $S$ .*

We will need the analogous result for cones, which we prove using the above theorem together with a theorem due to Hammer [8] bounding the ratio in which the centroid of a convex body can divide a chord of that body:

**Theorem 3.2** ([8]). *Let  $K$  be an  $n$ -dimensional convex body, and let  $p$  be the centroid of  $K$ . Then, for each chord  $[x, y]$  of  $K$  through  $p$ , the convex combination  $p = (1 - \mu)x + \mu y$  satisfies  $\frac{1}{n+1} \leq \mu \leq \frac{n}{n+1}$ .*

**Theorem 3.3.** *Let  $C \subset V$  be an  $n$ -dimensional cone, and let  $p \in \text{int}(C)$ . Then there exists a bounded section  $S$  of  $C$  such that  $p$  is the centroid of  $S$ .*

*Proof.* Fix  $\lambda > n + 1$ , and let  $H$  be a hyperplane such that  $C \cap H$  is bounded and  $C \cap (\lambda p - C)$  lies in a closed half-space bounded by  $H$ . Let  $K$  be the intersection of  $C$  with this closed half-space. Since  $K$  is a convex body, there exists a section  $S$  of  $K$  with centroid  $p$  by Theorem 3.1.

We claim that  $S$  does not intersect  $C \cap H$ . Suppose otherwise, and let  $[x, y]$  be a chord of  $S$  through  $p$  with  $x \in \partial C$  and  $y \in C \cap H$ . Let  $y' := x + \lambda(p - x)$ . Then  $y' = \lambda p - (\lambda - 1)x \in \partial(\lambda p - C)$  is the point where the ray from  $x$  through  $p$  meets  $\partial(\lambda p - C)$ . Since  $p = (1 - \frac{1}{\lambda})x + \frac{1}{\lambda}y'$ , and since  $y' \in [x, y]$ , we must have that  $p = (1 - \mu)x + \mu y$  for some  $\mu \leq 1/\lambda < \frac{1}{n+1}$ . Therefore, by Theorem 3.2,  $p$  is not a centroid of  $S$ , a contradiction.

Since  $S$  does not intersect  $C \cap H$ , it follows that  $S$  is a bounded section of  $C$  with centroid  $p$ , as desired.  $\square$

**Lemma 3.4.** *Let  $C$  be a cone with  $\dim C \geq 2$ . If  $p \in \text{int}(C)$  is the center of symmetry of a bounded section  $S$  of  $C$ , then  $\partial S = \partial C \cap \partial(2p - C)$ .*

*Proof.* Let  $\Gamma := \partial C \cap \partial(2p - C)$ . Suppose that  $\partial S \neq \Gamma$ . Then, by Lemma 2.2, there is a point  $x \in \partial S$  such that  $\mu x \in \Gamma$  for some  $\mu \neq 1$ . Let  $L$  be the 2-dimensional linear span of  $p$  and  $x$ . By Lemma 2.1,  $\partial S$  and  $\Gamma$  are both centrally symmetric about  $p$ . Inversion through  $p$  fixes  $L$ , so we must have that  $2p - x \in \partial S \cap L$  and  $2p - \mu x \in \Gamma \cap L$ . By applying Lemma 2.2 to the 2-dimensional cone  $C \cap L$ , we find that  $2p - x$  and  $2p - \mu x$  must be scalar multiples of each other. However, this is evidently not the case, since  $p$  and  $x$  form a basis for  $L$  and  $\mu \neq 1$ . Thus,  $\partial S = \Gamma$ , as claimed.  $\square$

It is now straightforward to prove Theorem 1.4 as a corollary of the above results.

*Proof of Theorem 1.4 (stated on p. 2).* Suppose that  $C$  is a finite-dimensional CSS cone. Let a point  $a \in \text{int}(C)$  be given, and let  $p := \frac{1}{2}a$ . By Theorem 3.3, there is a bounded section  $S$  of  $C$  with centroid  $p$ . By hypothesis,  $S$  is centrally symmetric. The centroid of a centrally symmetric convex body is the center of symmetry of the body, so  $p$  is the center of symmetry of  $S$ . By Lemma 3.4,  $\partial C \cap \partial(a - C) = \partial S \subset \text{aff}(S)$ , which implies that  $C$  is an FBI cone. Therefore, by Theorem 1.2,  $C$  is ellipsoidal.  $\square$

## REFERENCES

- [1] P. W. Aitchison, C. M. Petty, and C. A. Rogers, *A convex body with a false centre is an ellipsoid*, *Mathematika* **18** (1971), 50–59.
- [2] H. K. Brunn, *Über kurven ohne wendepunkte*, Habilitationsschrift, Ludwig-Maximilians-Universität München, 1889.
- [3] H. Busemann, *The geometry of geodesics*, Academic Press Inc., New York, N. Y., 1955.
- [4] A. Emch, *Some Properties of Closed Convex Curves in a Plane*, *Amer. J. Math.* **35** (1913), no. 4, 407–412.
- [5] P. R. Goodey and M. M. Woodcock, *Intersections of convex bodies with their translates*, *The geometric vein*, Springer, New York, 1981, pp. 289–296.
- [6] P. M. Gruber and T. Ódor, *Ellipsoids are the most symmetric convex bodies*, *Arch. Math. (Basel)* **73** (1999), no. 5, 394–400.
- [7] B. Grünbaum, *On some properties of convex sets*, *Colloq. Math.* **8** (1961), 39–42.
- [8] P. C. Hammer, *The centroid of a convex body*, *Proc. Amer. Math. Soc.* **2** (1951), 522–525.
- [9] T. Hausel, E. Makai, Jr., and A. Szűcs, *Polyhedra inscribed and circumscribed to convex bodies*, *Proceedings of the Third International Workshop on Differential Geometry and its Applications and the First German-Romanian Seminar on Geometry (Sibiu, 1997)*, vol. 5, 1997, pp. 183–190.
- [10] E. Morales-Amaya, *Secciones y proyecciones de cuerpos convexos*, B.S. thesis, Universidad Nacional Autónoma de México, 1998.
- [11] S. Olovjanischnikoff, *Ueber eine kennzeichnende Eigenschaft des Ellipsoides*, *Uchenye Zapiski Leningrad State Univ., Math. Ser.* **83(12)** (1941), 114–128.
- [12] C. A. Rogers, *Sections and projections of convex bodies*, *Portugal. Math.* **24** (1965), 99–103.
- [13] B. Solomon, *Surfaces with central convex cross-sections*, *Comment. Math. Helv.* **87** (2012), no. 2, 243–270.
- [14] H. Steinhaus, *Quelques applications des principes topologiques à la géométrie des corps convexes*, *Fund. Math.* **41** (1955), 284–290.

(Jesús Jerónimo-Castro) FACULTAD DE INGENIERIA, UNIVERSIDAD AUTÓNOMA DE QUERÉTARO, CERRO DE LAS CAMPANAS S/N, C.P. 76010, QUERÉTARO, MÉXICO

(Tyrrell B. McAllister) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WYOMING, LARAMIE, WY 82071, USA

*E-mail address*, Tyrrell B. McAllister: [tmcallis@uwyo.edu](mailto:tmcallis@uwyo.edu)